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Spinning test particle in Kalb–Ramond background

D. Maity[1,a], S. SenGupta[1,b], S. Sur[2,c]

[1] Department of Theoretical Physics, Indian Association for the Cultivation of Science, Calcutta – 700 032, India
[2] Institute of Physics, Bhubaneswar – 751005, India

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Abstract. In this work we explore the geodesic deviations of spinning test particles in a string inspired Einstein–Kalb–Ramond background. Such a background is known to be equivalent to a spacetime geometry with torsion. We have shown here that the antisymmetric Kalb–Ramond field has a significant effect on the geodesic deviation of a spinning test particle. A search for observational evidence of such an effect in astrophysical experiments may lead to a better understanding of the geometry of the background spacetime.

1 Introduction

String theory, since its emergence, is considered to be the most promising candidate for a consistent perturbative quantum theory of gravity. The search for the signature of string theory in the low energy world has intensified over the last few years to establish contact of this theory with the real world. The possible testing grounds are accelerator experiments and cosmological/astrophysical observations. The present work aims to search for such a stringy signal in an astrophysical observation through the study of geodesic deviations of spinning test particles. It is well known that the low energy limit of the gravity sector of string theory indeed reproduces the curved spacetime picture as proposed in Einstein's theory. However, the presence of the massless second rank antisymmetric tensor field (Kalb–Ramond field) [1] endows the background spacetime with torsion. Thus a string inspired background differs from Einstein's framework by a Cartan extension.

In Einstein's framework the dynamics of particles in curved spacetime has been an important subject of investigation. Since the early stages of the development of general relativity, the study of the motion of a test particle (i.e., a particle which is sufficiently small compared to other objects producing the field and which has a negligible influence on the field) in a curved background is of great importance. Studies of the dynamics of streams of cosmic particles in astrophysical/cosmological experiments reveal the important properties of the background spacetime. As is well known a test particle of the simplest type, i.e., one without any internal structure, has been shown to follow the so-called geodesics. Such test particles with single pole structures are referred to as "pole particles". However, a test particle can have a structure of its own, thereby giving rise to a non-vanishing spin-density for the particle. As such, its equation of motion can then depend on this structure. A test particle with such a *multipole* structure is expected to follow a trajectory that is different from that of the usual geodesic. A theory describing the motion of such "pole–multipole particles" has been developed initially by Papapetrou [2] and later on by Dixon [3] in an alternative approach. Subsequent applications of the Papapetrou formalism to the particular case of the motion in a static spherically symmetric Schwarzschild field has been carried out by Corinaldesi and Papapetrou [4] and since then there has been a growing interest in the study of the motion of spinning test particles under the influence of gravity. Most recently, there have been a plethora of works in the literature [5] that investigate the dynamics of spinning test particles in different sorts of background spacetimes.

In this work, we study the motion of spinning particles in a general static spherisymmetric spacetime in presence of the Kalb–Ramond (KR) field. It has been shown [6] in the context of string theory that the KR field, in general, has a gauge invariant coupling with the electromagnetic field and produces significant effects on many cosmological/astrophysical phenomena [7–10]. Extensive works have also been carried out [11,12] to explore the role of such an antisymmetric tensor background in the context of compact extra dimensional theories of Arkani-Hamed–Dimopoulous–Dvali (ADD) [13] and Randall–Sundrum (RS) [14] types. The observational possibilities of a stringy signal has also been discussed in several other works [15, 16]. Here we explore another possible experimental signature of such a string inspired background through the influence of KR field on the geodesics of spinning test particles. We show that such geodesics indeed differ from those observed in a pure Einstein background. An estimate of

[a] e-mail: tpdm@iacs.res.in
[b] e-mail: tpssg@iacs.res.in
[c] e-mail: sourav@iopb.res.in

this departure is made in terms of the KR field and spin of the test particle.

2 General equation of motion of spinning test particle

According to the formalism of Papapetrou [2], the trajectory of a spinning test particle in an arbitrary spacetime structure is shown to deviate from the usual geodesic and is described by

$$\frac{D}{D\tau}\left(m\ u^{\alpha}+u_{\beta}\frac{DS^{\alpha\beta}}{D\tau}\right)+\frac{1}{2}\ R^{\alpha}_{\mu\nu\lambda}\ u^{\nu}\ S^{\lambda\mu}=0, \quad (1)$$

where τ is the proper time, m and u^{α} are respectively the particle's mass and four-velocity, $S^{\alpha\beta}$ is the antisymmetric spin tensor of the particle, $R^{\alpha}_{\ \mu\nu\lambda}$ is the curvature tensor corresponding to the background field distribution on which the particle moves, and $D/D\tau$ denotes the covariant derivative along u^{α}:

$$\frac{DS^{\alpha\beta}}{D\tau}=\frac{\mathrm{d}S^{\alpha\beta}}{\mathrm{d}\tau}+\left(\Gamma^{\alpha}_{\mu\nu}\ S^{\mu\beta}+\Gamma^{\beta}_{\mu\nu}\ S^{\alpha\mu}\right)u^{\nu}, \quad (2)$$

$\Gamma^{\alpha}_{\mu\nu}$ being the usual Christoffel connections.

The spin of the particle evolves as

$$\frac{DS^{\alpha\beta}}{D\tau}+u^{\alpha}u_{\rho}\ \frac{DS^{\beta\rho}}{D\tau}-u^{\beta}u_{\rho}\ \frac{DS^{\alpha\rho}}{D\tau}=0. \quad (3)$$

For vanishing spin one can easily verify that the above trajectory equation (1) reduces to the usual geodesic equation.

The above equations are, however, not sufficient to determine all the unknowns. It may be noted that the number of independent equations determining the spin components is three, while the number of independent spin components is six. Therefore, to reduce the number of independent spin components one has to impose a suitable supplementary condition that specifies the line L that represents the motion of a "pole–dipole" particle inside the world tube of the particle [2]. The simplest supplementary condition suggested by Corinaldesi and Papapetrou [4] on investigating the motion of a spinning test particle in a static spherisymmetric spacetime has been

$$S^{0i}=0\ ;\quad i=1,2,3. \quad (4)$$

This condition, although not covariant, provides a very physical definition of the representing world line L of the spinning test particle. One can, in fact, show that if the coordinates of L are designated by X^{α} (which are functions of the proper time τ along L), then in the rest frame of the central attracting body each point $X\in L$ coincides with the center of mass of the particle. Other kinds of supplementary conditions can also be found in the literature [17]; however, we presently resort to the above condition owing to the physical relevance mentioned above.

Considering a general static spherically symmetric spacetime metric structure, viz.,

$$\mathrm{d}\tau^2=-\mathrm{e}^{\nu(r)}\mathrm{d}t^2+\mathrm{e}^{\lambda(r)}\mathrm{d}r^2+r^2\left(\mathrm{d}\vartheta^2+\sin^2\vartheta\mathrm{d}\varphi^2\right), \quad (5)$$

one can obtain, on using the above supplementary equation (4), the explicit form of the spin evolution equations (3):

$$\dot{S}^{12}+\left(\frac{\lambda'}{2}+\frac{1}{r}-\frac{\nu'}{2}\right)\dot{r}\ S^{12}+r\mathrm{e}^{-\lambda}\ \sin^2\vartheta\ \dot{\varphi}\ S^{23}+\cos\vartheta\ \sin\vartheta\ \dot{\varphi}\ S^{31}=0, \quad (6a)$$

$$\dot{S}^{23}+\left(\frac{\nu'}{2}-\frac{1}{2}\right)\dot{\varphi}\ S^{12}+\left(\frac{2\dot{r}}{r}+\cot\vartheta\ \dot{\vartheta}\right)S^{23}+\left(\frac{\nu'}{2}-\frac{1}{r}\right)\dot{\vartheta}\ S^{31}=0, \quad (6b)$$

$$\dot{S}^{31}-\cot\vartheta\ \dot{\varphi}\ S^{12}+r\mathrm{e}^{-\lambda}\ \dot{\vartheta}\ S^{23}+\left[\left(\frac{1}{r}+\frac{\lambda'}{2}-\frac{\nu'}{2}\right)\dot{r}+\cot\vartheta\ \dot{\vartheta}\right]S^{31}=0, \quad (6c)$$

where the overhead dot indicates differentiation with respect to τ and a prime denotes differentiation with respect to r. The equations of motion (1) for the test particle now take the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(m+m_s)\,\dot{t}\right]+(m+m_s)\,\Gamma^0=0, \quad (7a)$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(m+m_s)\,\dot{r}\right]+(m+m_s)\,\Gamma^1+r\mathrm{e}^{-\lambda}\left(\frac{\lambda'}{2}+\frac{\nu'}{2}\right)\left(S^{12}\ \dot{\vartheta}-S^{31}\ \sin^2\vartheta\ \dot{\varphi}\right)=0, \quad (7b)$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(m+m_s)\,\dot{\vartheta}\right]+(m+m_s)\,\Gamma^2+r\left(\frac{\lambda'\nu'}{4}-\frac{\nu'^2}{4}-\frac{\nu''}{2}-\frac{\lambda'}{2r}\right)S^{12}+r\mathrm{e}^{-\lambda}\ \sin^2\vartheta\ \dot{\varphi}\left(\frac{\nu'}{2}+\frac{\mathrm{e}^{\lambda}}{r}-\frac{1}{r}\right)S^{23}=0, \quad (7c)$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(m+m_s)\,\dot{\varphi}\right]+(m+m_s)\,\Gamma^3-r\mathrm{e}^{-\lambda}\ \dot{\vartheta}\left(\frac{\nu'}{2}+\frac{\mathrm{e}^{\lambda}}{r}-\frac{1}{r}\right)S^{23}-\dot{r}\left(\frac{\lambda'\nu'}{4}-\frac{\nu'^2}{4}-\frac{\nu''}{2}-\frac{\lambda'}{2r}\right)S^{31}=0, \quad (7d)$$

where $\Gamma^{\mu}\ \equiv\ \Gamma^{\mu}_{\nu\lambda}\ u^{\nu}\ u^{\lambda}$ with components

$$\Gamma^0=\nu'\ \dot{r}\ \dot{t},$$
$$\Gamma^1=\frac{\lambda'}{2}\ \dot{r}^2-r\mathrm{e}^{-\lambda}\ \dot{\vartheta}^2-r\mathrm{e}^{-\lambda}\ \sin^2\vartheta\ \dot{\varphi}^2+\mathrm{e}^{(\nu-\lambda)}\ \nu'\ \dot{t}^2,$$
$$\Gamma^2=\frac{2}{r}\ \dot{r}\ \dot{\vartheta}-\sin\vartheta\ \cos\vartheta\ \dot{\varphi}^2$$
$$\Gamma^3=\frac{2}{r}\ \dot{r}\ \dot{\varphi}+2\ \cot\vartheta\ \dot{\vartheta}\ \dot{\varphi}. \quad (8)$$

Here m is the particle's mass and the quantity m_s is defined by

$$m_s=\frac{r^2}{2}\ \nu'\ \left(\sin^2\vartheta\ \dot{\varphi}\ S^{31}-\dot{\vartheta}\ S^{12}\right). \quad (9)$$

m_s can be viewed as an effective mass originating from the spin–orbit coupling and $(m+m_s)$ is the total effective mass.

In what follows, we shall be investigating whether the above two sets of equations (6) and (7) admit any solution representing a motion on a plane passing through the central body, of course, with the view that both the mass and the spin of the test particle have exceedingly small effects on the background spacetime. As usual, one can take this plane without any loss of generality as the equatorial plane $\vartheta = \pi/2$ making the simple choice

$$S^{31} \neq 0\,; \qquad S^{12} = S^{23} = 0, \tag{10}$$

whence $m_s = (r^2/2)\ \nu'\ \sin^2\vartheta\ \dot\varphi\ S^{31}$. This leads to the following set of equations of motion:

$$\dot S^{31} + \left(\frac{1}{r} - \frac{\nu' - \lambda'}{2}\right) \dot r\ S^{31} = 0, \tag{11a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(m+m_s)\,\dot t\right] + (m+m_s)\,\nu'\ \dot r\ \dot t = 0, \tag{11b}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(m+m_s)\,\dot\varphi\right] + (m+m_s)\,\frac{2\dot r\dot\varphi}{r} + \left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + \frac{\lambda'}{2r}\right)\dot r\ S^{31} = 0, \tag{11c}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(m+m_s)\,\dot r\right] + (m+m_s)\left\{\frac{\dot r^2\lambda'}{2} - r\mathrm{e}^{-\lambda}\ \dot\varphi^2 + \frac{\nu'\dot t^2}{2}\ \mathrm{e}^{(\nu-\lambda)}\right\} - r\mathrm{e}^{-\lambda}\ \left(\frac{\nu'+\lambda'}{2}\right)\ \dot\varphi\ S^{31} = 0. \tag{11d}$$

Equation (11b) at once gives the integral of energy

$$\mathrm{e}^{\nu}\ \dot t\,(m+m_s) = E\ \text{(const.)}, \tag{12}$$

while (11a) stands as a first integral of spin only:

$$r\ S^{31}\ \mathrm{e}^{(\lambda-\nu)/2} = K\ \text{(const.)}. \tag{13}$$

Using (12) and (13) and noticing that

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} = \dot t^2\ \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \ddot t\ \frac{\mathrm{d}}{\mathrm{d}t},$$

the variable τ can be eliminated from (11c) and (11d), whence we obtain

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} + \left(\frac{\lambda'}{2} - \nu'\right)\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - r\mathrm{e}^{-\lambda}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 + \frac{\nu'}{2}\ \mathrm{e}^{(\nu-\lambda)} - \frac{K}{E}\ \mathrm{e}^{\frac{3(\nu-\lambda)}{2}}\ \frac{(\nu'+\lambda')}{2}\ \frac{\mathrm{d}\varphi}{\mathrm{d}t} = 0, \tag{14a}$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} + \left(\frac{2}{r} - \nu'\right)\frac{\mathrm{d}r}{\mathrm{d}t}\ \frac{\mathrm{d}\varphi}{\mathrm{d}t} + \frac{K}{E}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + \frac{\lambda'}{2r}\right)\mathrm{e}^{\frac{(3\nu-\lambda)}{2}}\ \frac{1}{r}\ \frac{\mathrm{d}r}{\mathrm{d}t} = 0. \tag{14b}$$

3 Spinning test particle trajectory in static spherisymmetric Einstein–Kalb–Ramond spacetime

Following the formalism in [6] the solutions for the metric coefficients in a general static spherical symmetric spacetime involving the KR field have been obtained in [18, 9]:

$$\mathrm{e}^{\nu(r)} = 1 - \frac{r_{\mathrm{S}}}{r} + b\left[\frac{r_{\mathrm{S}}}{6r^3} + \frac{r_{\mathrm{S}}^2}{6r^4} + \frac{3r_{\mathrm{S}}(r_{\mathrm{S}}^2 - b/2)}{20r^5} + \cdots\right],$$
$$\mathrm{e}^{-\lambda(r)} = 1 - \frac{r_{\mathrm{S}}}{r} + b\left[\frac{1}{r^2} + \frac{r_{\mathrm{S}}}{2r^3} + \frac{r_{\mathrm{S}}^2}{3r^4} + \frac{r_{\mathrm{S}}(r_{\mathrm{S}}^2 - b/6)}{4r^5} + \cdots\right], \tag{15}$$

where $r_{\mathrm{S}} = 2GM$ is the Schwarzschild radius and the constant b is a measure of the strength of the KR field (which has a natural interpretation in the form of a background torsion). The parameter b can be negative or positive depending the nature of the torsion–KR field coupling constant within a minimal coupling prescription as has been mentioned in [19]. Accordingly, torsion may exhibit a repulsive (anti-gravitating) or an attractive character. In the special case of vanishingly small gravitating mass $M = 0$, i.e., $r_{\mathrm{S}} = 0$, the above solutions reduce to the simple closed-form structures [18]

$$\mathrm{e}^{\nu(r)} = 1\,; \qquad \mathrm{e}^{-\lambda(r)} = 1 - \frac{b}{r^2}. \tag{16}$$

Depending on positive or negative value of b, these represent a "wormhole" of throat radius $\sqrt{|b|}$ or a "naked singularity" at $r = 0$.

We first study the dynamics of the spinning test particle in the special case $r_{\mathrm{S}} = 0$ and then follow this up with a more rigorous investigation in the general case $r_{\mathrm{S}} \neq 0$.

3.1 An otherwise empty spacetime in presence of Kalb–Ramond field

In the case $r_{\mathrm{S}} = 0$, when the Einstein–KR spacetime is otherwise empty, the equations of motion (14) for the spinning test particle reduce to

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} + \frac{\lambda'}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - r\mathrm{e}^{-\lambda}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 - \frac{K}{E}\ \frac{\lambda'}{2}\ \mathrm{e}^{-3\lambda/2}\ \frac{\mathrm{d}\varphi}{\mathrm{d}t} = 0, \tag{17a}$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} + \frac{2}{r}\ \frac{\mathrm{d}r}{\mathrm{d}t}\ \frac{\mathrm{d}\varphi}{\mathrm{d}t} + \frac{K}{E}\ \frac{\lambda'}{2\,r^2}\ \mathrm{e}^{-\lambda/2}\ \frac{\mathrm{d}r}{\mathrm{d}t} = 0. \tag{17b}$$

Equation (17b) for φ yields the integral of angular momentum

$$r^2\ \frac{\mathrm{d}\varphi}{\mathrm{d}t} - \frac{K}{E}\ \mathrm{e}^{-\lambda/2} = I_s\ \text{(const.)}. \tag{18}$$

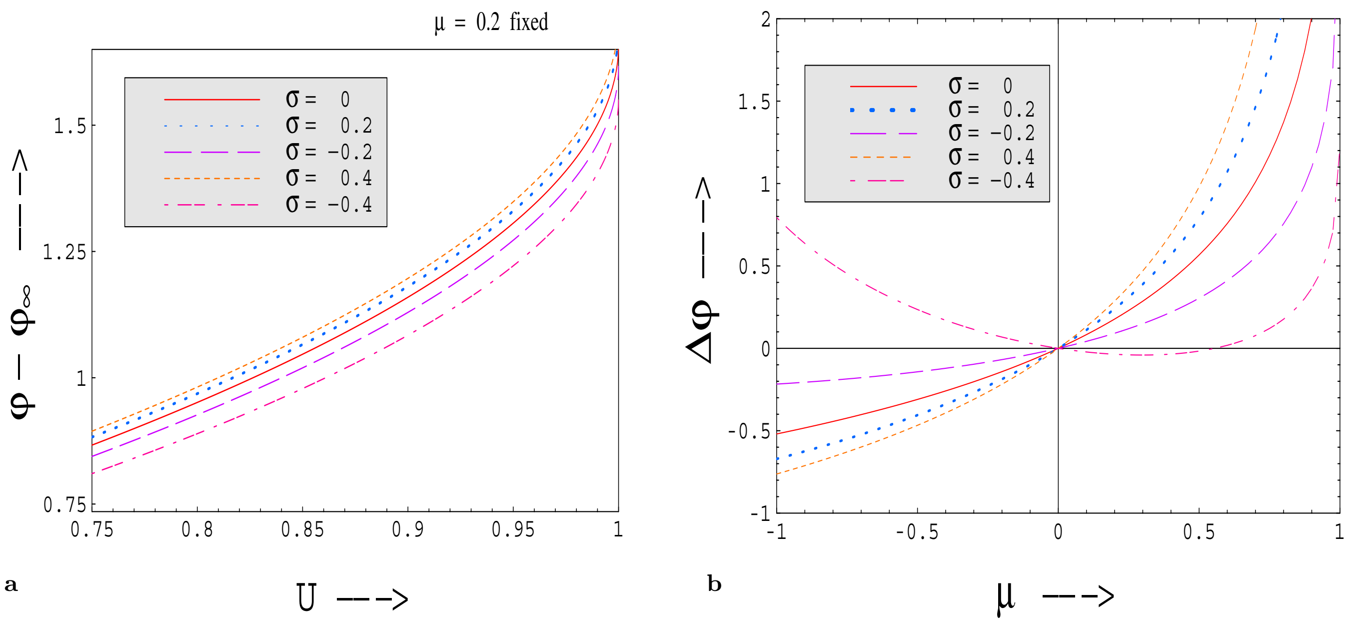


Fig. 1. a Variation of $[\varphi(r) - \varphi_\infty]$ with $U = r_0/r$ for a characteristically chosen fixed value ($= 0.2$) of the parameter $\mu = b/r_0^2$ and a range of values of the parameter σ from -0.4 to 0.4. In order to achieve appreciable deviations the plots have been exaggerated for higher values of U (close to unity). **b** Plots of the angle of bending $\Delta\varphi$ as a function of μ for parametric values of σ ranging from -0.4 to 0.4

Equation (17a) for r then reduces to

$$\frac{\mathrm{d}\varphi}{\mathrm{d}U} = \frac{\mathrm{e}^{\lambda/2} + \sigma}{\sqrt{(1 + \sigma\, \mathrm{e}^{-\lambda_0/2})^2 - U^2\, (1 + \sigma\, \mathrm{e}^{-\lambda/2})^2}}, \qquad (19)$$

where we have used the dimensionless independent variable $U = r_0/r$, r_0 being the distance of closest approach of the spinning test particle towards the center of force; $\lambda_0 \equiv \lambda(r_0)$ and as such $\mathrm{e}^{-\lambda(U)} = 1 - \mu U^2$, where $\mu = b/r_0^2$ is the dimensionally scaled KR parameter. $\sigma = K/(EI_s)$ is a dimensionless parameter that contains the integrals of energy, spin and orbital angular momentum which describe the trajectory of the test particle.

As we are primarily interested in obtaining the equation of the trajectory of the spinning test particle at distances far away from the center of force ($r \gg r_0$, i.e., $U \ll 1$) where the KR field is sufficiently weak ($|b| \ll r^2$ for all $r \geq r_0$), we make a large-distance approximation up to order U^2 and write the above equation in the integral form as

$$\int \mathrm{d}\phi \approx \frac{1}{1+\sigma} \int \frac{\mathrm{d}u}{\sqrt{\delta^2 - U^2}} \left\{1 + \sigma + \frac{\mu U^2}{2} + \mathcal{O}\left(\mu U^2\right)^2\right\}; \quad U = \frac{r_0}{r} \ll 1, \qquad (20)$$

where

$$\delta = \frac{1}{1+\sigma}\left[1 + \sigma\left\{1 - \frac{\mu}{2} + \mathcal{O}\left(\mu^2\right)\right\}\right]; \quad |\mu| = \frac{|b|}{r_0^2} \ll 1. \qquad (21)$$

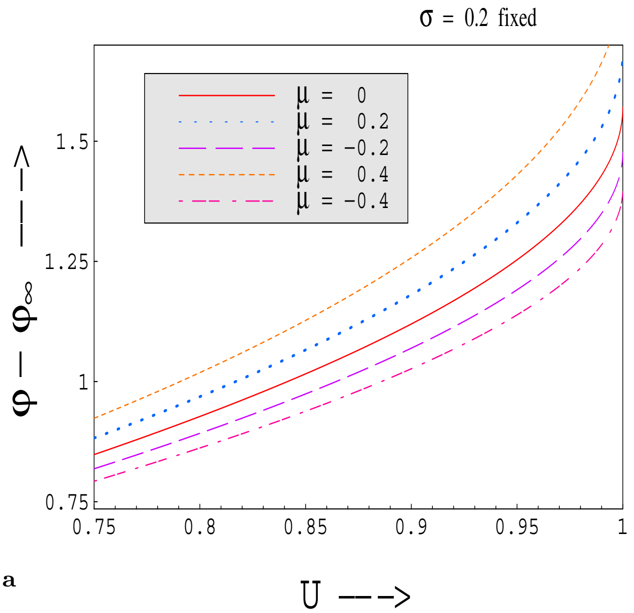
The solution of (20) is given by

$$\varphi - \varphi_\infty = \sin^{-1}\frac{U}{\delta} + \frac{\mu\, \delta^2}{4\,(1+\sigma)}\left[\sin^{-1}\frac{U}{\delta} - \frac{U}{\delta}\, \sqrt{\delta^2 - U^2}\right] + \mathcal{O}(\mu U^2)^2, \qquad (22)$$

where φ_∞ is the value of φ in the asymptotic limit $r \to \infty$. The second term on the right hand side of (22) gives a measure of the bending of the trajectory of the spinning test particle (time-like or null) in the Einstein–KR spacetime. This departure from the straight line motion which is a characteristic in Minkowski spacetime (in absence of the KR field) signals the exclusive effect of the antisymmetric background on the geodesic and may be called as KR-lensing. The spin of the particle described by the parameter σ does not contribute to the bending independently, but is coupled with the KR parameter $\mu = b/r_0^2$ in an otherwise empty spacetime. In fact, the presence of the KR field makes the effect of the particle spin on the trajectory deviation perceptible even when the gravitating mass is zero. The change inflicted by the spin parameter σ on the expression for $(\varphi - \varphi_\infty)$ for a pure KR background (i.e., for $\sigma = 0$ – the result in [9]) is given to the leading order in $\mu\sigma$ as

$$\frac{\sigma\mu}{4}\left[\left(2 - \sqrt{1 - U^2}\right)\, U - \sin^{-1} U\right].$$

Now, instead of performing an approximate integration of the equation of motion (19) for $U \ll 1$, one can also solve (19) numerically, for characteristically chosen numerical values of the parameters σ and μ. The amounts



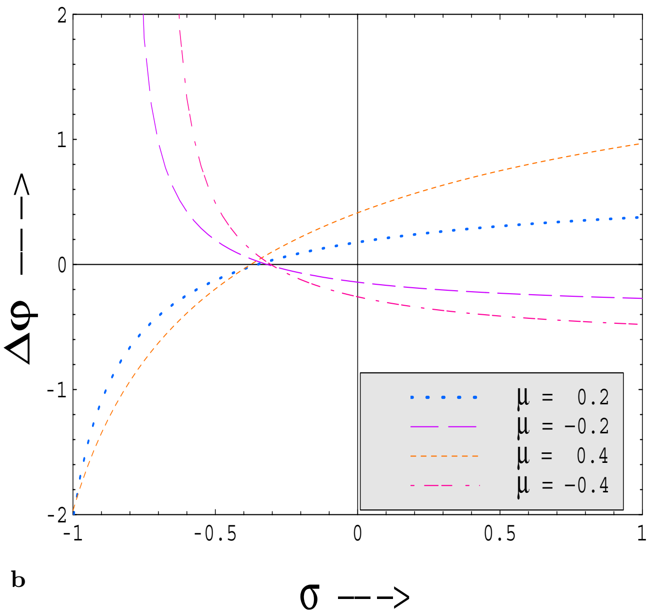


Fig. 2. a Variation of $[\varphi(r)-\varphi_\infty]$ with $U=r_0/r$ for a characteristically chosen fixed value $(=0.2)$ of the parameter σ and a range of values of the parameter μ from -0.4 to 0.4. In order to achieve appreciable deviations the plots have been exaggerated for higher values of U (close to unity). **b** Plots of the angle of bending $\Delta\varphi$ as a function of σ for parametric values of σ ranging from -0.4 to 0.4. For $\mu=0$, however, $\Delta\varphi=0$

of trajectory deviation of the spinning test particles in such a numerical evaluation are depicted in Figs. 1a and 2a, while the variations of the angle of bending, viz., $\Delta\varphi = 2|\varphi-\varphi_\infty|-\pi$ as a function of μ (for fixed σ) or as a function of σ (for fixed μ) are shown respectively in Figs. 1b and 2b.

While the above results are achieved in an idealized situation where a vanishingly small gravitating mass M (and hence r_S) is considered, in the following section we investigate the motion of spinning test particles in the more realistic scenario of a general static spherical spacetime background in presence of the KR field, i.e with both b and $r_S \neq 0$.

3.2 A general static spherically symmetric spacetime in presence of Kalb–Ramond field

In the general static spherically symmetric spacetime ($r_S \neq 0$) with the metric in the form (5), we have the differential equations (14a) and (14b) for r and φ. Equation (14b) yields the general integral of angular momentum given by

$$r^2\,\mathrm{e}^{-\nu}\,\frac{\mathrm{d}\varphi}{\mathrm{d}t}-\frac{K}{E}\,\mathrm{e}^{(\nu-\lambda)/2}\left(1-\frac{r\nu'}{2}\right)=I_g\ (\text{const.}). \quad (23)$$

Considering the particle's spin to be small we obtain the equation of the orbit in an power series expansion of the redefined dimensionless spin parameter $\sigma=K/(EI_g)$ as

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = r^4\,\mathrm{e}^{-\lambda} \times\left[\frac{1}{r_0^2}-\frac{1}{r^2}+\frac{1}{I_g^2}\left(\mathrm{e}^{-\nu}-\mathrm{e}^{-\nu_0}\right) + 2\sigma\left(\frac{1}{r_0^2}\,\mathrm{e}^{(\nu_0-\lambda_0)/2}-\frac{1}{r^2}\,\mathrm{e}^{(\nu-\lambda)/2}\right)+\mathcal{O}\left(\sigma^2\right)\right] \Bigg/ \left[1+\sigma\left(1-\frac{r\nu'}{2}\right)\mathrm{e}^{(\nu-\lambda)/2}\right]^2. \quad (24)$$

Due to the extreme complexity of this equation in the general static spherisymmetric Einstein–KR spacetime with solutions for e^{ν} and $\mathrm{e}^{-\lambda}$ as given in (15), we focus on the rather simplified scenario, that of the dynamics of null (massless) spinning particles in such a spacetime.

Now, resorting to the limit $\sigma\to 0$, we have $I_g \to I_0 = r^2\mathrm{e}^{-\nu}\dot{\varphi}$ and the above equation (24) can be recast [20, 21] in the form

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = r^2\,\mathrm{e}^{-\lambda(r)}\,\left(\frac{r^2}{D^2}\,\mathrm{e}^{-\nu(r)}-1\right), \quad (25)$$

where $D=I_0$ is the impact parameter for the null particles. The solution is given in the form of a quadrature [20, 21],

$$\varphi(r)-\varphi_\infty = \int_r^\infty \frac{\mathrm{d}r'}{r'}\,\mathrm{e}^{\lambda(r')/2}\,\left(\frac{r'^2}{D^2}\,\mathrm{e}^{-\nu(r')}-1\right)^{-1/2}. \quad (26)$$

At the distance of closest approach (r_0) to the center of force, $\mathrm{d}r/\mathrm{d}\varphi|_{r=r_0}=0$, whence (25) gives

$$D=I_0=r_0\,\mathrm{e}^{-\nu_0/2}, \quad (27)$$

where $\nu_0 \equiv \nu(r_0)$.

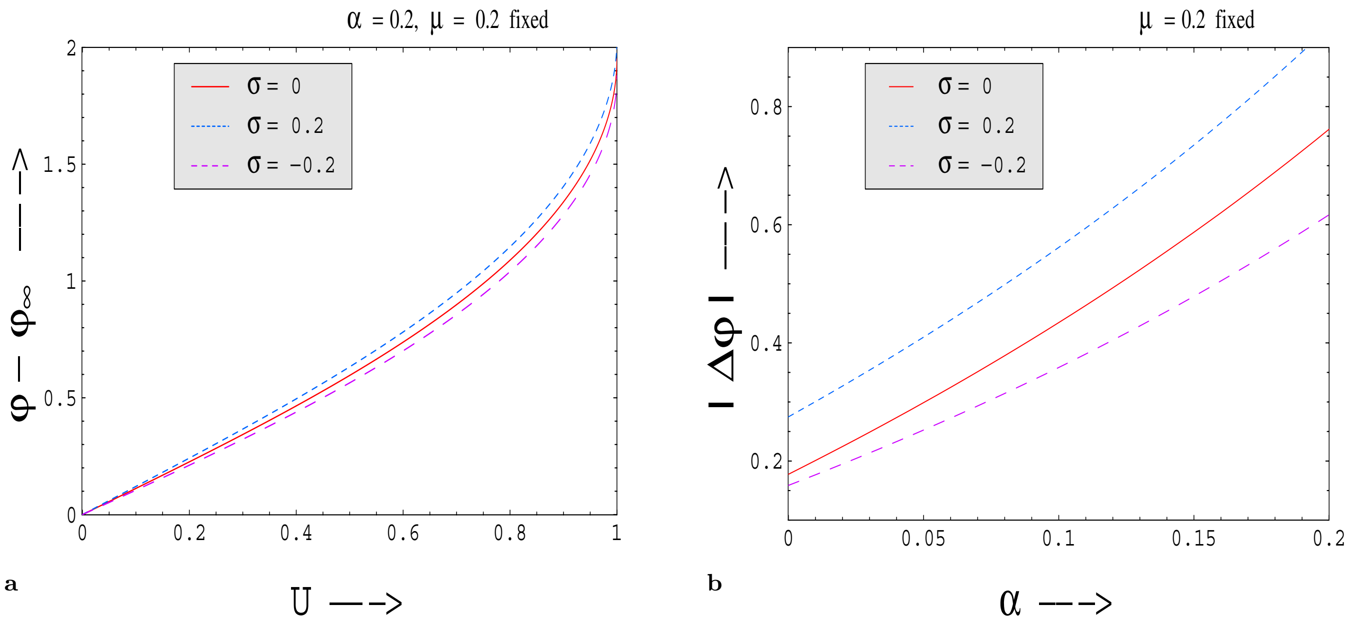


Fig. 3. a Variation of $[\varphi(r) - \varphi_\infty]$ with $U = r_0/r$ for characteristically chosen fixed values of the parameters $\alpha(= 0.2)$ and $\mu(= 0.2)$ and three values $(= 0, 0.2$ and $-0.2)$ of the parameter σ. **b** Plots of the magnitude of angle of bending $\Delta\varphi$ as a function of α for a fixed value of $\mu(= 0.2)$ and three parametric values $(= 0, 0.2$ and $-0.2)$ of σ

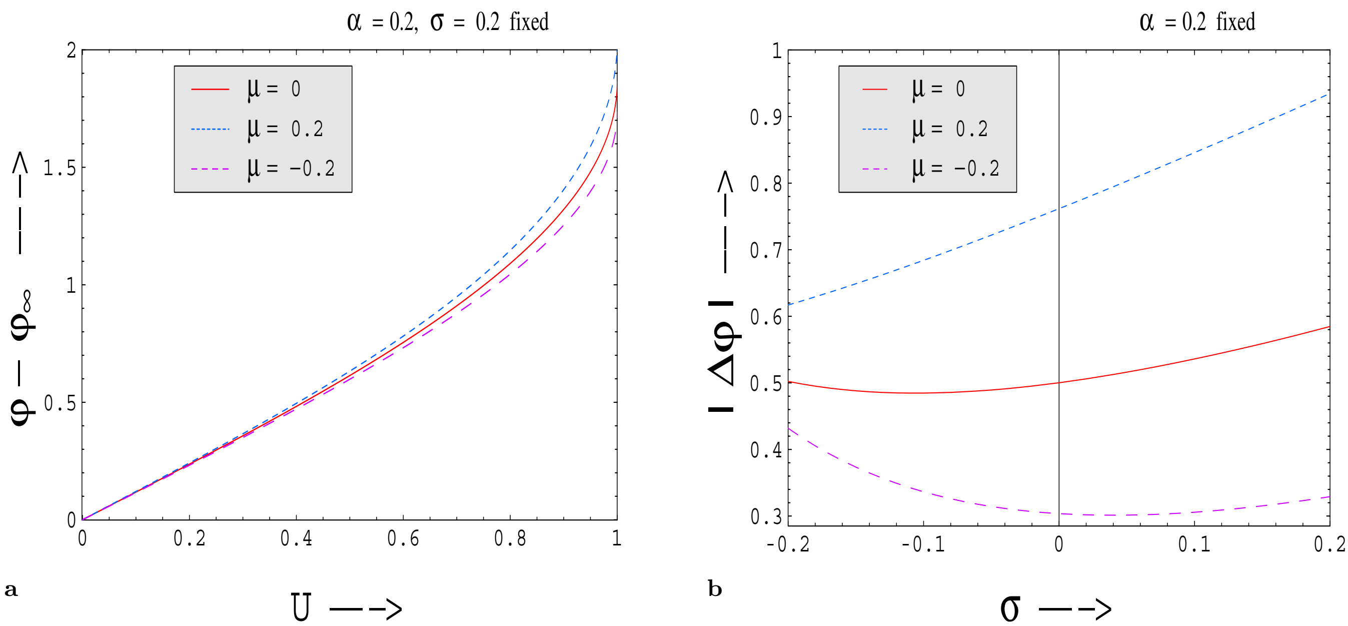


Fig. 4. a Variation of $[\varphi(r) - \varphi_\infty]$ with $U = r_0/r$ for characteristically chosen fixed values of the parameters $\alpha(= 0.2)$ and $\sigma(= 0.2)$ and three values $(= 0, 0.2$ and $-0.2)$ of the parameter μ. **b** Plots of the magnitude of angle of bending $\Delta\varphi$ as a function of σ for a fixed value of $\alpha(= 0.2)$ and the three parametric values $(= 0, 0.2$ and $-0.2)$ of μ

For non-vanishing σ, (23) for the integral of angular momentum can be rewritten as

$$I_g = \frac{I_0}{1 + \sigma\left(1 - \frac{r\nu'}{2}\right) \mathrm{e}^{(\nu-\lambda)/2}}\,; \qquad I_0 = r^2\, \mathrm{e}^{-\nu}\, \dot{\varphi}. \quad (28)$$

Using (27) and (28) and changing the independent variable to $U \equiv r_0/r$, we finally write the equation for the orbit as

$$\frac{\mathrm{d}\varphi}{\mathrm{d}U} = \pm \mathrm{e}^{\lambda/2} F(U) \Big/ \left(1 - U^2 + \left(\mathrm{e}^{\nu_0 - \nu} - 1\right) F^2(U) + 2\sigma G(U) + \mathcal{O}\left(\sigma^2\right)\right)^{1/2}, \quad (29)$$

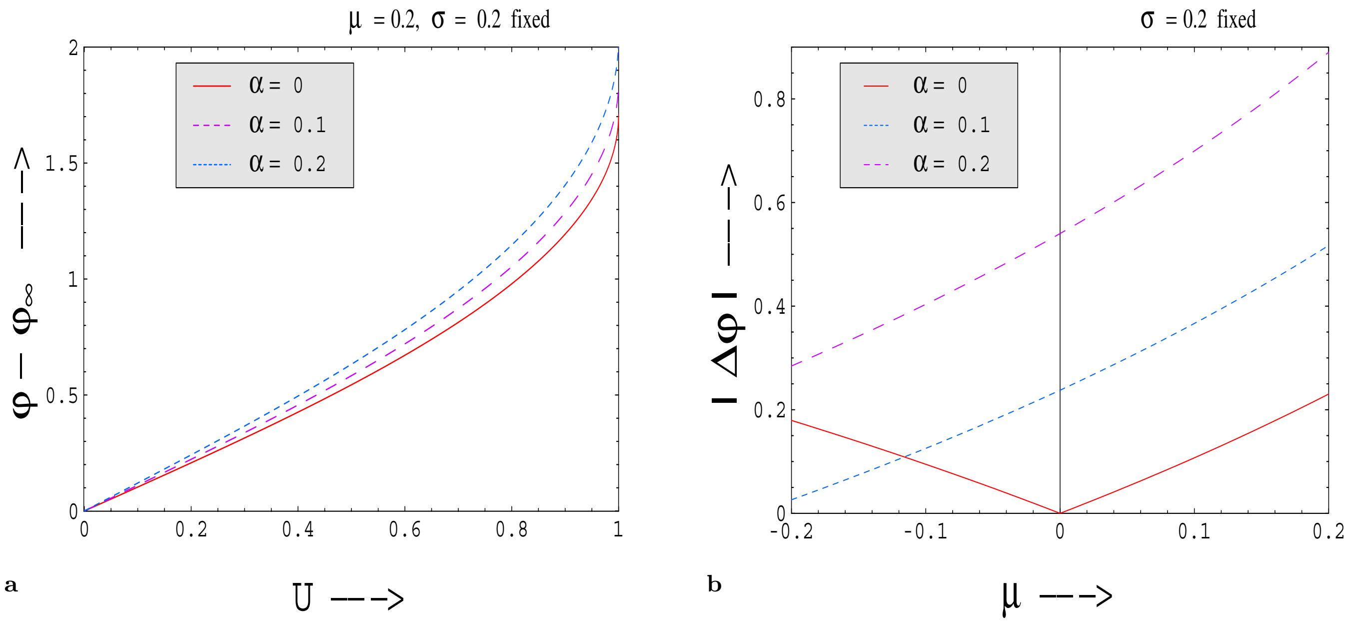


Fig. 5. **a** Variation of $[\varphi(r) - \varphi_\infty]$ with $U = r_0/r$ for characteristically chosen fixed values of the parameters $\mu(= 0.2)$ and $\sigma(= 0.2)$ and three values $(= 0, 0.1$ and $0.2)$ of the parameter α. **b** Plots of the magnitude of angle of bending $\Delta\varphi$ as a function of μ for a fixed value of $\sigma(= 0.2)$ and the three parametric values $(= 0, 0.1$ and $0.2)$ of α

where

$$F(U) = 1 + \sigma\left(1 + \frac{U}{2}\frac{\mathrm{d}\nu}{\mathrm{d}U}\right)\mathrm{e}^{(\nu-\lambda)/2} \; ;$$
$$G(U) = \mathrm{e}^{(\nu_0-\lambda_0)/2} - U^2\mathrm{e}^{(\nu-\lambda)/2}. \qquad (30)$$

It is easy to check that for $r_S = 0$, up to order σ^2, (29) reduces exactly to (19), the approximate analytic solution of which is given in (22). For $r_S \neq 0$, however, in absence of such analytic solution, we resort to numerical techniques to solve (29) up to order σ^2. Now, in a general static spherisymmetric Einstein–KR spacetime, one's prime interest is in the determination of effects of the particle's spin as well as those of the KR field on standard astrophysical phenomena. For this purpose, we need to study the trajectory deviation of spinning particles in the gravitational field of non-compact objects whose mean radii are much larger than the Schwarzschild radius r_S. In other words, the region $r \gg r_S$ is of particular relevance in the context of KR and particle-spin effects. Incidentally, both the metric coefficients e^ν and $\mathrm{e}^{-\lambda}$ given in (15) are convergent for $r \gg r_S$, provided the torsion (or, the KR field strength) is small, i.e., $|b|/r^2 \ll 1$, in that domain. Since there is no experimental signature directly in favour of torsion till now, it is reasonable to consider torsion to be weak, at least in the region $r \geq r_0$, with r_0 (the distance of closest approach towards the center of force) much larger than the Schwarzschild radius. As such, we consider the magnitude of the dimensionless KR measure $\mu = b/r_0^2$ to be much smaller than unity. Dropping terms of order quadratic or more in r_S/r and b/r^2 we can approximately write the solutions (15) in terms of the dimensionally scaled radial coordinate U and the KR parameter μ as

$$\mathrm{e}^{\nu(U)} = 1 - \alpha U \; ; \qquad \mathrm{e}^{-\lambda(U)} = 1 - \alpha U - \mu U^2, \qquad (31)$$

where $\alpha = r_S/r_0$. For these solutions of the metric coefficients, we obtain the numerical solutions of (29) for various sets of values of the dimensionless parameters α, μ and σ. The corresponding bending angle of trajectories $(\Delta\varphi = 2|\varphi - \varphi_\infty| - \pi)$ have been computed in various situations. The plots of $(\varphi - \varphi_\infty)$ as a function of U as well as those of $\Delta\varphi$ as a function of α or σ or μ have been shown in Figs. 3–5.

Now to estimate the KR (or torsion) measure μ (and also the spin parameter σ) we follow the standard solar system analysis as in [9]. For the bending of light near the sun, the impact parameter r_0 is roughly of the order of the solar radius R_s and the parameter α is estimated to be

$$\alpha = \frac{r_S}{r_0} = \frac{2GM_s}{c^2 R_s} \qquad (32)$$

where M_s is the solar mass. Plugging in the standard values for M_s and R_s [21,22] we find α to be extremely small $(\sim 5 \times 10^{-6})$. The spin parameter $\sigma = K/(EI_g)$ can be estimated as follows: the value of the spin angular momentum K for photons is $\hbar$ and the energy $E = hc/\lambda$, where h is the Planck's constant, $\hbar = h/(2\pi)$ and c is the speed of light. Therefore, $K/E = \lambda/(2\pi c)$ which is numerically $\sim 10^{-15}$ for visible radiation with $\lambda \sim 5000\,\mathring{A}$. From (28), the constant I_g can be given in terms of the parameters r_0, α and μ as

$$I_g = \frac{r_0}{\sqrt{1-\alpha}} - \frac{K}{E}\left(1 - \frac{3\alpha}{2}\right)\sqrt{1 - \frac{\mu}{1-\alpha}}. \qquad (33)$$

With the small estimates of α and K/E shown above, one can approximately write $I_g = r_0 \sim R_s$, which essentially implies an extremely small value of the dimensionless spin parameter σ ($\sim 10^{-23}$). Neglecting the small estimates for α and σ, and integrating (29) and (30), we finally compute the leading contribution of the KR field to the amount of bending of light trajectories as

$$\Delta\varphi|_{\mathrm{KR}} = \frac{\pi\mu}{4}. \tag{34}$$

Using the error bars for the standard deflection of light measurements for the sun [22] the parameter $\mu = b/r_0^2 \approx b/R_s^2$ turns out to be approximately of the order of 10^{-6}.

4 Conclusion

We have clearly demonstrated the influence of a string inspired KR background on the geodesics of spinning test particles. We have shown how the geodesic is modified by various factors, namely the KR field strength, the spin of the particle and the gravitating mass. The dependence of the geodesics on these factors have been exhibited graphically. We hope that an accurate determination of the geodesic deviation of various cosmic particles in astrophysical experiments would be able to pinpoint the presence or absence of antisymmetric tensor field in the background spacetime leading to a much better understanding of the background spacetime geometry. In addition, any evidence of the presence of such a massless antisymmetric tensor field in the background may be looked upon as indirect evidence of a string inspired low energy world.

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